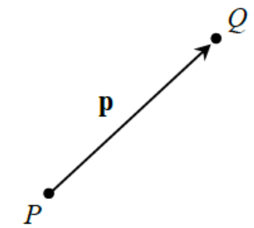
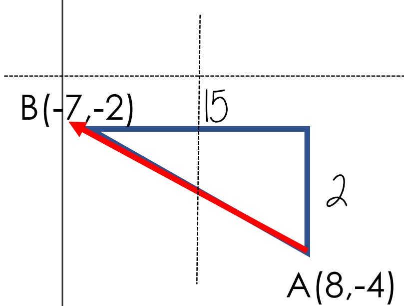
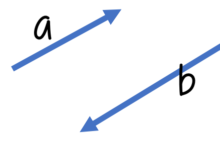
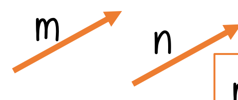
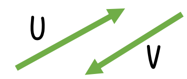
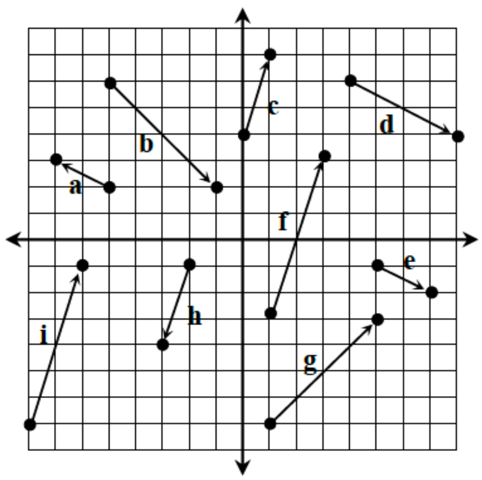
Vectors

Main Ideas/Questions	Notes/Examples
VECTOR	A quantity that has both magnitude (size) AND direction.
GEOMETRICAL Representation	A vector can be represented geometrically using a **directed line segment:** • P is the initial point, or **tail**. • Q is the terminal (end) point, or **tip**. 
NAMING VECTORS	• Vectors are denoted using the $\rightarrow$ symbol. • Vector PQ above can be named as $\overrightarrow{PQ}$, $\vec{p}$, or $\mathbf{p}$ (a boldface lowercase letter).
MAGNITUDE	Given a vector $\mathbf{v}$ with initial point (x_1, y_1) and terminal point (x_2, y_2), the magnitude of $\mathbf{v}$, $\lVert\mathbf{v}\rVert$, can be found using the distance formula: $$\lVert\mathbf{v}\rVert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
 ** Think of your vector as being on the coordinate plane. If you created a right triangle, the vector would be the hypotenuse.	**Find the magnitude of each vector.** **1.** $\overrightarrow{AB}$ with $A(8, -4)$ and $B(-7, -2)$ $\lVert AB\rVert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $\sqrt{(-15)^2 + (2)^2}$ $\boxed{\sqrt{229} \text{ or } 15.13}$ **2.** $\overrightarrow{RS}$ with $R(-3, 10)$ and $S(5, 6)$ **3.** $\overrightarrow{PQ}$ with $P(-8, 6)$ and $Q(2, -9)$ **4.** $\overrightarrow{EF}$ with $E(-1, -5)$ and $F(-6, -7)$
ZERO VECTOR	A vector with a magnitude (size) of zero!

TYPES OF VECTORS

PARALLEL VECTORS	EQUIVALENT VECTORS	OPPOSITE VECTORS
Same or opposite direction, but not necessarily the same magnitude.	Same magnitude and direction.	Same magnitude but opposite directions. The opposite of vector **a** is written as **-a**.
 a‖b	 m=n	 v= -u

Use the graph below to classify each pair of vectors.



5. **a** and **e**	opposite
6. **i** and **f**	equal
7. **d** and **e**	parallel
8. **b** and **g**	none
9. **c** and **h**	opposite

Proving Vectors are EQUIVALENT

You can show two vectors are equivalent if they have:

- the **same magnitude** (use the distance formula) BOTH!
- the **same direction** (use the slope formula)

Determine whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent given their intial and terminal points.

10. $\overrightarrow{AB}$ with $A(-6, -7)$ and $B(-2, 3)$; $\overrightarrow{CD}$ with $C(2, -10)$ and $D(6, 0)$

$$\|\overrightarrow{AB}\| = \sqrt{(-2+6)^2+(3+7)^2} = \sqrt{116} = 2\sqrt{29}$$

$$AB: m = \frac{3+7}{-2+6} = \frac{10}{4} = \frac{5}{2}$$

$$\|\overrightarrow{CD}\| = \sqrt{(6-2)^2+(0+10)^2} = \sqrt{116} = 2\sqrt{29}$$

$$CD: m = \frac{0+10}{6-2} = \frac{10}{4} = \frac{5}{2}$$

$\boxed{\text{Equivalent}}$

Vectors: at-home work

Directions: Find the magnitude of each vector. Distance formula!

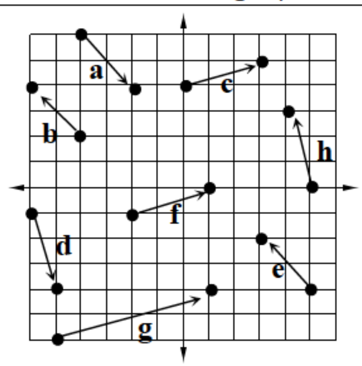
1. $\overrightarrow{VW}$ with $V(3, -2)$ and $W(5, 2)$

2. $\overrightarrow{LM}$ with $A(6, 2)$ and $B(-2, -7)$

3. $\overrightarrow{AB}$ with $A(2, 3)$ and $B(-7, 1)$

4. $\overrightarrow{EF}$ with $E(-8, 8)$ and $F(-10, -2)$

Directions: Use the graph below to classify each pair of vectors.



5. a and **e**	**6. c** and **f**
7. f and **g**	**8. b** and **e**
9. c and **d**	**10. d** and **h**

Directions: Use the distance and slope formula to determine whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent.

11. $\overrightarrow{AB}$ with $A(4, 8)$ and $B(6, -9)$; $\overrightarrow{CD}$ with $C(-3, 11)$ and $D(-1, -6)$

12. $\overrightarrow{AB}$ with $A(1, 2)$ and $B(-1, -5)$; $\overrightarrow{CD}$ with $C(-8, -1)$ and $D(-6, -8)$

Directions: Give the component form, magnitude, and direction angle for each vector.

13. p = <-3, -4>

Distance formula

$\|p\| = \sqrt{(-3)^2 + (-4)^2}$

$= \sqrt{25}$

$= 5$

$\tan\theta = \frac{4}{3}$

$\theta = 233.13°$

$\tan\theta = \frac{y}{x}$

Magnitude:	Direction Angle:
5	233.13^0

14. **d** = <-1, 7>

Magnitude:	Direction Angle:

15. $\overrightarrow{CD}$ with $C(-8, 7)$ and $D(2, -5)$

Component Form:	Magnitude:	Direction Angle:
$< \Delta x, \Delta y >$	dist. form.	$\tan\theta$

16. $\overrightarrow{PQ}$ with $P(-3, 3)$ and $Q(-11, -1)$

Component Form:	Magnitude:	Direction Angle:

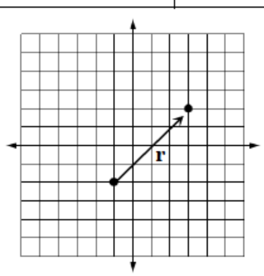
17. $\overrightarrow{JK}$ with $J(-6, 9)$ and $K(-5, 14)$

Component Form:	Magnitude:	Direction Angle:

18. $\overrightarrow{XY}$ with $X(4, -3)$ and $Y(-3, -1)$

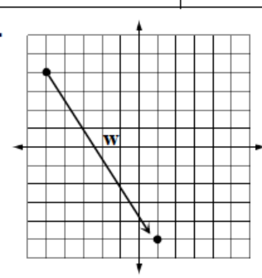
Component Form:	Magnitude:	Direction Angle:

19.



Component Form:	Magnitude:	Direction Angle:

20.

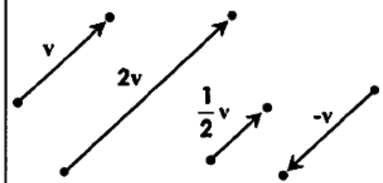


Component Form:	Magnitude:	Direction Angle:

Vectors with "Math Stuff"

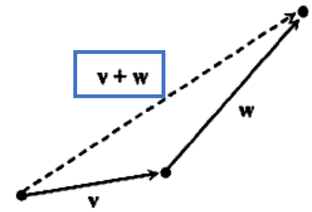
MULTIPLYING *by a Scalar*

A vector can be multiplied by a real number k, called a scalar. This will change the magnitude of the vector. If k is negative, the vector will reverse to the opposite direction. The figure to the right shows various scalars of vector **v**.



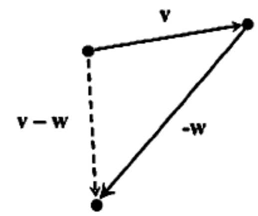
ADDING *Vectors*

A geometric representation of the addition of two vectors, **v** and **w**, is shown to the right. The vectors are positioned so that the tip of **v** coincides with the tail of **w**. The sum of the vectors **v + w**, called the **resultant**, extends from the tail of **v** to the tip of **w**.



SUBTRACTING *Vectors*

The difference of two vectors **v** and **w** is defined as **v − w = v + (-w)**. A geometric representation of the difference **v − w** is shown to the right using the same "tip-to-tail" model shown above.



VECTOR OPERATION *Rules*

If **v** = <a_1, b_1> and **w** = <a_2, b_2> are vectors and k is a real number, then:

SCALAR MULTIPLCATION	$k\mathbf{v} = \langle ka_1, kb_1 \rangle$
ADDITION	$\mathbf{v} + \mathbf{w} = \langle a_1 + a_2, b_1 + b_2 \rangle$
SUBTRACTION	$\mathbf{v} - \mathbf{w} = \langle a_1 - a_2, b_1 - b_2 \rangle$

Examples

Find each of the following for a = ⟨−6, 2⟩, b = ⟨1, 7⟩, and c = ⟨-4, 5⟩.

1. 2c

< 2(-4), 2(5) >

$= \boxed{\langle -8, 10 \rangle}$

2. a + b

a:<-6, 2> b:<1, 7>

a+b=<-6+1 , 2+7>

$= \boxed{\langle -5, 9 \rangle}$

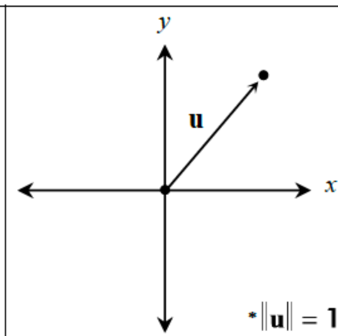
3. c − b

4. a − 3c

UNIT VECTORS

- A vector with a **magnitude of 1** is called a **unit vector.**
- Given any non-zero vector $\mathbf{v}$, a unit vector $\mathbf{u}$ in the same direction of $\mathbf{v}$ can be written using the following rule:

$$\boxed{u = \frac{1}{\|v\|} v}$$



$$u = \frac{1}{\|\mathbf{v}\|} \cdot \frac{\mathbf{v}}{1}$$

magnitude & direction

magnitude

> You can think of unit vectors a bit like finding an equivalent fraction. It has the same value, just a different way of writing it!

Examples

1) Calculate magnitude $\|\mathbf{v}\|$
2) Multiply by $\frac{1}{\mathbf{v}}$

Find a unit vector u with the same direction as the given vector.

7. $\mathbf{v} = \langle -5, 12 \rangle$

$\|v\| = \sqrt{(-5)^2 + 12^2} = 13$

$u = \frac{1}{13} \langle -5, 12 \rangle$

$\boxed{\left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle}$

8. $\mathbf{n} = \langle 9, 12 \rangle$

$\|n\| = \sqrt{9^2 + 12^2} = 15$

$u = \frac{1}{15} \langle 9, 12 \rangle$

$\boxed{\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle}$

9. $\mathbf{p} = \langle -4, -1 \rangle$

10. $\mathbf{k} = \langle 16, -8 \rangle$

11. $\overline{AB}$ with $A(0, -7)$ and $B(6, -1)$

1) $\Delta x, \Delta y$ → $AB = \langle 6-0, -1+7 \rangle = \langle 6, 6 \rangle$

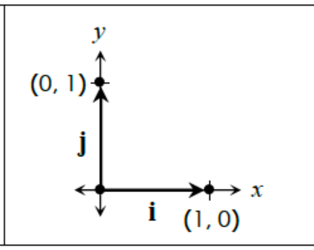
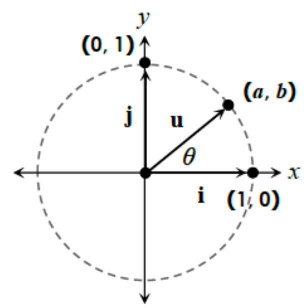
2) Calculate magnitude (dist. form) → $\|AB\| = \sqrt{6^2 + 6^2} = \sqrt{72} = 6\sqrt{2}$

3) Multiply by $\frac{1}{\mathbf{v}}$ → $u = \frac{1}{6\sqrt{2}} \langle 6, 6 \rangle$

4) That = u (your "equivalent fraction") → $= \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \boxed{\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle}$

12. $\overline{PQ}$ with $P(-1, 4)$ and $Q(8, -2)$

Vectors in Terms of Trigonometry

Main Ideas/Questions	Notes/Examples
STANDARD UNIT VECTORS	The **standard unit vectors, i** and **j,** are positioned along the x- and y-axis, and defined as: **i** = <1,0> Direction on the **x-axis** **j** = <0,1> Direction on the **y-axis** 
LINEAR COMBINATIONS of i and j	Any vector **v** = <*a*, *b*> can be written in the form ai + bj. This is called a **linear combination** of vectors **i** and **j**. Follow the steps below to prove **v = <*a*, *b*> = *a*i + *b*j.** Rewrite the vector v=⟨2,5⟩ as a linear combination of i and j. v = 2i + 5j
TRIGONOMETRIC FORMS of a Unit Vector	• If **u** is a unit vector, then the terminal point of **u** lies on the Unit circle. • Therefore, **u** = <*a*, *b*> can be written as: $u = \langle \cos\theta, \sin\theta \rangle$, or as the linear combination $u = \langle \cos\theta\, i, \sin\theta\, j \rangle$. 

Remember!

Any point (x,y) on the unit circle can be represented by (cos, sin)

TRIGONOMETRIC FORMS of any Vector	Because every vector **v** is the product of its magnitude $\lVert \mathbf{v} \rVert$ and its corresponding unit vector **u**, we can write **v** in the following forms: $\mathbf{v} = \lVert \mathbf{v} \rVert \mathbf{u} = \lVert v \rVert \langle \cos\theta, \sin\theta \rangle$, $\langle \lVert v \rVert \cos\theta, \lVert v \rVert \sin\theta \rangle$, or as the linear combination $\langle \lVert v \rVert (\cos\theta)\, i, \lVert v \rVert (\sin\theta)\, j \rangle$.

EXAMPLES

Given: magnitude and direction
Use:
$\|v\| \langle v\cos\theta, v\sin\theta \rangle$

Directions: Find the component form of each vector with the given magnitude and direction angle.

5. $\|\mathbf{v}\| = 8,\ \theta = 60°$	**6.** $\|\mathbf{w}\| = 20,\ \theta = 225°$
$v = \langle 8\cos 60,\ 8\sin 60 \rangle$ $= \langle 8(\frac{1}{2}),\ 8(\frac{\sqrt{3}}{2}) \rangle$ $= \boxed{\langle 4,\ 4\sqrt{3} \rangle}$	$w = \langle 20\cos 225,\ 20\sin 225 \rangle$ $= \langle 20(-\frac{\sqrt{2}}{2}),\ 20(-\frac{\sqrt{2}}{2}) \rangle$ $= \boxed{\langle -10\sqrt{2},\ -10\sqrt{2} \rangle}$

Given: magnitude
Use:
- dist. formula to find $\|v\|$
- $\tan^{-1}$ to find θ (direction)
- then, plug into

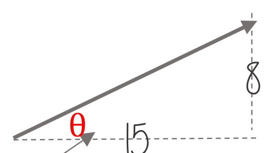
$\|v\| \langle v\cos\theta, v\sin\theta \rangle$

Directions: Write each vector in trigonometric form.

9. $\mathbf{v} = \langle 15,\ 8 \rangle$	**10.** $\mathbf{m} = \langle -3\sqrt{2},\ -\sqrt{6} \rangle$
$\|v\| = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$ $\tan\theta = \frac{8}{15};\quad \theta = 28.07°$ $v = \langle 17\cos 28.07°,\ 17\sin 28.07° \rangle$ or $v = 17\langle \cos 28.07°,\ \sin 28.07° \rangle$	
	12. $\mathbf{r} = -2\mathbf{i} + 2\mathbf{j}$ $\|r\| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ $\tan\theta = -1;\quad \theta = 135°$ $r = \langle 2\sqrt{2}\cos 135°,\ 2\sqrt{2}\sin 135° \rangle$ or $r = 2\sqrt{2}\langle \cos 135°,\ \sin 135° \rangle$

Why $\tan^{-1}$?

- Vector gives you the points in 2 directions,
- If you draw it, it creates a right triangle
- Tangent = Opposite/adjacent



v= $\langle 15, 8 \rangle$

Given: two points
Use:
- Find vector $(\Delta x, \Delta y)$
- dist. formula to find $\|v\|$
- $\tan^{-1}$ to find θ (direction)
- then, plug into

$\|v\| \langle v\cos\theta, v\sin\theta \rangle$

13. $\overline{AB}$ with $A(-1, 7)$ and $B(-9, 3)$	**14.** $\overline{YZ}$ with $Y(3, -2)$ and $Z(5, -8)$
$\vec{AB} = \langle -8, -4 \rangle$ $\|\vec{AB}\| = \sqrt{(-8)^2 + (-4)^2} = \sqrt{80} = 4\sqrt{5}$ $\tan\theta = \frac{1}{2};\quad \theta = 206.56°$ $\vec{AB} = \langle 4\sqrt{5}\cos 206.56°,\ \frac{1}{2}\sin 206.56 \rangle$ or $\vec{AB} = 4\sqrt{5}\langle \cos 206.56°,\ \sin 206.56° \rangle$	

Finding Your Way

Main Ideas/Questions	Notes/Examples

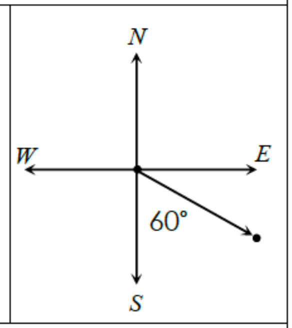
BEARINGS
& Direction

Note: When a degree measure is given with no additional directions, it is assumed to be a **true bearing**.

The **direction** of a vector can also be given as a **bearing**. Bearings are frequently given in application problems.

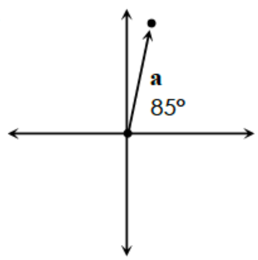
QUADRANT BEARING: The acute angle between a vector and a north-south line, or y-axis. The vector to the right is 60° east of south and is written as **S 60° E**.

TRUE BEARING: The angle measured clockwise from the north to the vector. The true bearing of the vector to the right is **120°**. True bearings are written with three digits, so a true bearing of 15° is written as 015°.



For each vector, give the quadrant bearing and the true bearing.

1.



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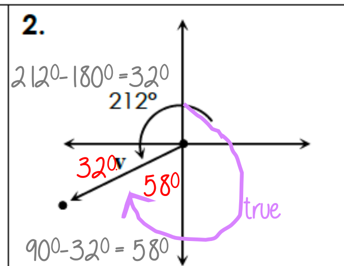

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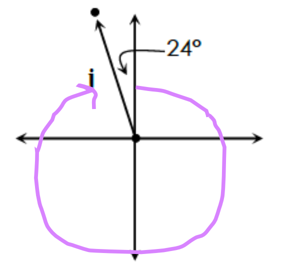
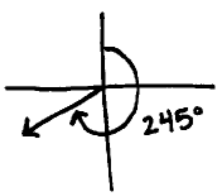


acute angle

three digits!!

	1.	2.	3.
Quadrant bearing:	N 5° E	S 58° W	S 58° W
True bearing:	005°	238°	

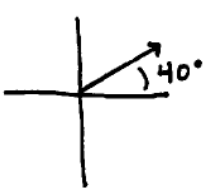


An airplane is flying at a bearing of 245° at 550 mph. Give the velocity of the airplane as a vector in component form.

$$V = \langle 550 \cos 205°, 550 \sin 205° \rangle$$

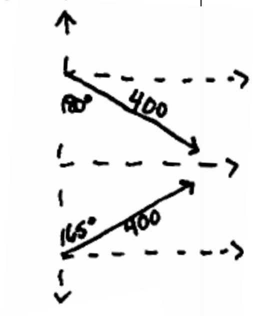
$$V = \langle -498.47, -232.44 \rangle$$

A man pushes a lawnmower with a force of 25 pounds uphill at an angle of 40°. Give the force exerted on the lawnmower as a vector in component form.



RESULTANT FORCE

Applications



If 2 or more forces are acting on an object, then the resultant force is the sum of the 2 vectors

7. Two tugboats are pushing a barge. If Tugboat A is traveling in the direction of S 80° E, Tugboat B is traveling in the direction of N 65° E, and both boats are pushing with a force of 400 pounds, find the magnitude and direction (as a quadrant bearing) of the resultant force on the barge.

$v_1 = \langle 400\cos 350°,\ 400\sin 350° \rangle$

$v_2 = \langle 400\cos 25°,\ 400\sin 25° \rangle$

$v_1 + v_2 = \langle 756.45,\ 99.59 \rangle$

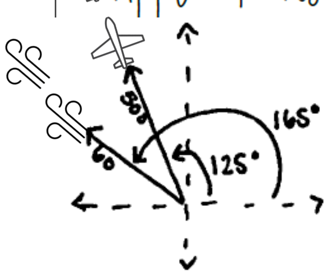
$\|v_1 + v_2\| = \sqrt{756.45^2 + 99.59^2} = \boxed{762.98 \text{ pounds}}$

$\tan\theta = \dfrac{99.59}{756.45}$; $\theta = 7.5° \rightarrow \boxed{\text{N } 82.5° \text{ E}}$

8. Two forces, with magnitudes of 85 pounds and 120 pounds, are acting on an object at angles of 35° and 140°, respectively, with the positive x-axis. Find the magnitude and direction (to the positive x-axis) of the resultant force.

RESULTANT VELOCITY

Applications



10. An airplane is traveling at a speed of 500 miles per hour at a bearing of 325°. Once the airplane reaches a certain point, it encounters a wind velocity of 60 miles per hour in the direction of N 75° W. Find the resultant speed and direction (as a true bearing) of the airplane.

$v_1 = \langle 500\cos 125°,\ 500\sin 125° \rangle$

$v_2 = \langle 60\cos 165°,\ 60\sin 165° \rangle$

$v_1 + v_2 = \langle -344.74,\ 425.11 \rangle$

$\|v_1 + v_2\| = \sqrt{(-344.74)^2 + 425.11^2} = \boxed{547.32 \text{ mph}}$

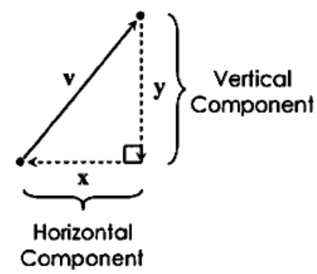
$\tan\theta = -\dfrac{425.11}{344.74}$; $\theta = 320.96°$

11. A baseball player runs forward at 15 feet per second and throws a baseball at a velocity of 80 feet per second at an angle of 20° with the horizontal. What is the resultant speed and direction of the throw?

Using Trig to Find Missing Components

HORIZONTAL & VERTICAL Components

- A vector can be **resolved** into a horizontal component and a vertical component.
- The horizontal and vertical components are also called **rectangular components**.
- Because the rectangular components form a right triangle, you can use trgionometric ratios to find their magnitudes.



14. Danika is pulling a sled with a force of 400 newtons using a string. If the string makes a 35° angle with the ground, find the magnitudes of the horizontal and vertical components.

$\cos 35° = \frac{|x|}{400}$

$|x| = 400 \cdot \cos 35°$

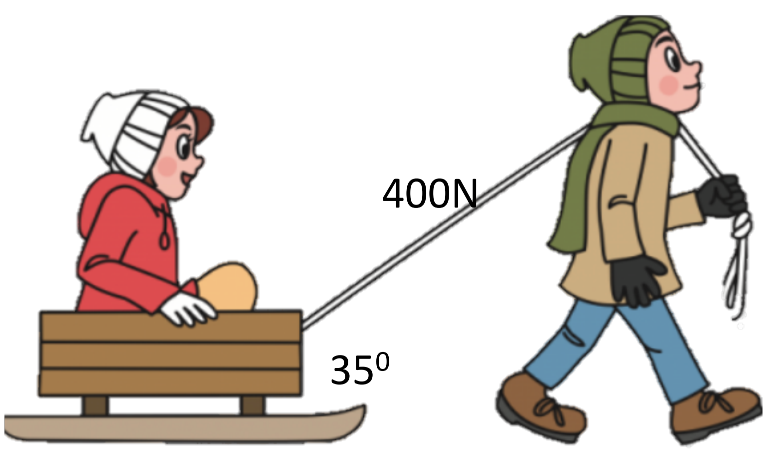
$|x| = 327.66$

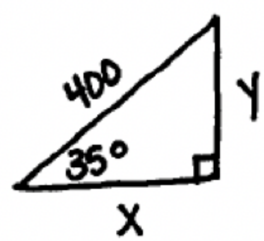
$\sin 35° = \frac{|y|}{400}$

$|y| = 400 \cdot \sin 35°$

$|y| = 229.43$

$\boxed{\text{Horizontal} = 327.66 \text{ pounds, Vertical} = 229.43 \text{ pounds.}}$





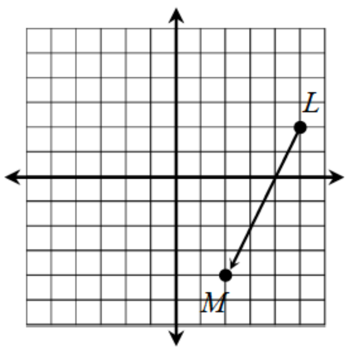
Putting it all Together

For each vector, give its (a) component form, (b) magnitude, and (c) direction angle.

1. $\overrightarrow{AB}$ with $A(-7, 4)$ and $B(-1, -4)$

2. $\overrightarrow{RS}$ with $R(4, -3)$ and $S(2, 9)$

3.



1. a) ____________

b) ____________

c) ____________

2. a) ____________

b) ____________

c) ____________

3. a) ____________

b) ____________

c) ____________

Find each of the following for r = <-6, -4>, s = <-2, 7>, and t = <5, -1>.

4. $3\mathbf{t} - \mathbf{r}$

5. $\frac{5}{2}\mathbf{r} + 4\mathbf{s}$

6. Find a unit vector **u** in the same direction as **v** = <8, -4>.

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

Give each vector as a linear combination in terms of unit vectors i and j.

7. **c** = <-1, -3>

8. $\overrightarrow{JK}$ with $J(-6, 1)$ and $K(-2, -8)$

Write the vector in component form given its magnitude and direction angle.

9. $\|v\| = 14$; $\theta = 135°$

10. $\|k\| = 3.5$; $\theta = 210°$

9. ______________

10. ______________

Give the trigonometric form of each vector.

11. $\mathbf{p} = \langle 2\sqrt{6}, -2\sqrt{2} \rangle$

12. $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$

11. ______________________________

12. ______________________________

13. ______________________________

13. $\overrightarrow{XY}$ with $X(1, 5)$ and $Y(-1, -9)$

14. Two construction workers are lifting a beam using ropes. Worker A pulls with a force of 500 newtons at an angle of 40° with the positive x-axis. Worker B is on the opposite side pulling with a force of 380 newtons at an angle of 115° with the positive x-axis. Find the **(a)** magnitude and **(b)** direction angle of the resultant force.

15. A boat is sailing at a bearing of 140° with a speed of 26 mph. It hits a current of 8 mph at a bearing of 195°. Find the resultant **(a)** magnitude and **(b)** direction (as a true bearing) of the boat.

14. a) ______________

b) ______________

15. a) ______________

b) ______________

16. a) ______________

b) ______________

16. Josh is swimming in a lake at a speed of 4 feet per second in the direction of N 24° W. Find the magnitude of the **(a)** horizontal component and **(b)** vertical component.